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An embedding theorem for finitely generated free metabelian groups

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Abstract

Using a construction due to Baumslag [2], we prove that a finitely generated free metabelian group can be embedded in a finitely presented metabelian group that is also residually nilpotent.

1. Introduction

In 1973 Gilbert Baumslag [2] proved that a finitely generated metabelian group can be embedded in a finitely presented metabelian group. This suggests the question as to whether a finitely generated metabelian group with a specific property can be embedded in a finitely presented metabelian group with the same property.

Here we will concern ourselves with *residual nilpotence*. We recall that a group H is residually nilpotent if $\bigcap_{i=1}^{\infty} \gamma_i H = 1$, where $\gamma_i H$ denotes the ith term of the lower central series of H. Our initial objective was to prove that if H is a finitely generated residually nilpotent metabelian group, then H can be embedded in a finitely presented metabelian group that is residually nilpotent. Here we shall prove this assertion in the case where H is a finitely generated free metabelian group.

We perhaps should point out that it is known that a finitely generated free metabelian group is residually nilpotent. (See, for example, [7].) Indeed, Gruenberg [3] proved that, in particular, the wreath product of two finitely generated free abelian groups is residually nilpotent. Since finitely generated free metabelian groups can be embedded in the wreath product of two finitely generated free abelian groups [6], it follows that they too are residually nilpotent. This was proved first by Magnus [5]. Here we note that it follows from Gruenberg's theorem that the wreath product of two infinite cyclic groups is residually nilpotent. This wreath product is a subgroup of a finitely presented metabelian group. The construction due to Baumslag [2] for obtaining such a finitely presented metabelian group involves the selection of a monic

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polynomial f in one variable of degree at least 1 with integer coefficients and constant term 1. If $f = 1 + t$, we obtain the group treated in an earlier paper by Baumslag [1]. Calling this group E, a computation shows that in its lower central series, $\gamma_i E = E'$, the derived group of E, for $i \geq 2$. That is, the group determined by $f = 1 + t$ is not residually nilpotent. On the other hand, the selection $f = 1 + t + t^2$ leads to a group which is residually nilpotent.

We make use of this observation to prove our main result.

Theorem. *A finitely generated free metabelian group can be embedded in a finitely presented metabelian group that is also residually nilpotent.*

2. Preliminaries

We first show that a finitely generated free metabelian group can be embedded in the wreath product of two finitely generated free abelian groups. As usual, a group G is termed free metabelian if there exists an absolutely free group F such that $G \simeq F/F''$. We will make use of the following theorem of Magnus [6], which appears here as Lemma 2.1. We write $H \wr K$ for the standard restricted wreath product of the groups H and K.

Lemma 2.1. *Let F be a free group, freely generated by the elements x_i, $i \in I$, and let $R \trianglelefteq F$. Let A be a free abelian group, freely generated by the elements a_i, $i \in I$. Then, $F/R' \hookrightarrow A \wr F/R$, where the embedding is given by the mapping $x_i R' \mapsto a_i(x_i R)$, $i \in I$.*

Letting $R = F'$, Lemma 2.1 implies that

$$F/F'' \hookrightarrow A \wr F/F'.$$

F/F' is free abelian of the same rank as F. Thus, for a free metabelian group G of rank n, we have

$$G \hookrightarrow A \wr T,$$

where A and T are free abelian of rank n. Let W denote $A \wr T$. According to Gruenberg [3], W is residually nilpotent.

As proved by Baumslag [2], a free metabelian group of rank n can be embedded in a finitely presented metabelian group. Our aim is to show that the latter group can be chosen to be residually nilpotent.

3. The embedding of W in a finitely presented metabelian group

In order to prove our main result, we follow the construction due to Baumslag [2]. Let B denote the base group of $W = A \wr T$, with A and T free abelian of rank n. And, for any group H, let $\mathbb{Z}H$ denote the integral group ring of H. We note the following simple lemma.

Lemma 3.1. *Let A be a free abelian group of finite rank and let T be a finitely generated abelian group. Then, for $W = A \wr T$, the base group B is a free $\mathbb{Z}T$-module.*

Proof. Recall that B is the restricted direct product of the conjugates $t^{-1}At = A^t$ of A by the elements $t \in T$. Since B is an abelian normal subgroup of W, we can view B as a $\mathbb{Z}(W/B)$-module. Identifying W/B with T, we then think of B as a $\mathbb{Z}T$-module. If A is free abelian on $\{a_1, a_2, \dots, a_n\}$, then it is clear, writing B additively, that B is a free $\mathbb{Z}T$-module on $\{a_1, a_2, \dots, a_n\}$. That is,

$$B = \bigoplus_{i=1}^{n} \mathbb{Z}Ta_i. \qquad \square$$

We need the usual notion of rings and modules of fractions. (See, for example, the discussion in [2]). Let R be a commutative ring with unity, let S be a multiplicatively closed subset of R containing unity, and let M be an R-module. With the usual definitions of addition and multiplication, the sets of "fractions" RS^{-1} and MS^{-1} become, respectively, a ring and an RS^{-1}-module.

Returning to $W = A \wr T$, with A and T free abelian of rank n, we select the following set of polynomials in $\mathbb{Z}T$:

$$F = \{f_i = 1 + t_i + t_i^2 \colon 1 \le i \le n\}.$$

Let S be the multiplicatively closed set generated by $F \cup \{1\}$. Exactly as in Baumslag's paper [2], we consider the multiplicative subgroup V of the field of fractions of $\mathbb{Z}T$ generated by $T, f_1, f_2, \dots, f_n$:

$$V = gp(T, f_1, f_2, \dots, f_n).$$

Since B is a free $\mathbb{Z}T$-module (Lemma 3.1), it follows that BS^{-1} is a free $\mathbb{Z}T \cdot S^{-1}$-module. (See, for example, [8].) Hence, V is a subgroup of the multiplicative group of units of the ring BS^{-1}. We form the split extension E of BS^{-1} by V:

$$E = BS^{-1} \rtimes V.$$

An explicit presentation for the group E is given in [2], and it is shown there, as well, that $W = A \wr T$ is embedded in E. Thus, our free metabelian group G, embedded in W (see Section 2), is also embedded in E. The group E is a finitely presented metabelian group. It is precisely this group E that we will show is residually nilpotent.

4. Sufficient condition for residual nilpotence

Lemmas 4.1 and 4.2 will enable us to produce the lower central series of $E = BS^{-1} \rtimes V$.

Let A be a commutative ring with unity and let V be a subgroup of the multiplicative group of units of A. Let A^+ denote the additive group of A and let Aut A^+ denote the group of automorphisms of A^+. Then we can define an action θ of V on A^+, i.e., a homomorphism:

$$\theta: V \to \operatorname{Aut} A^+.$$

Let H be the semidirect product of A^+ by V with this action:

$$H = A^+ \rtimes V.$$

We identify the elements a and $(1,a)$ for all $a \in A^+$, and similarly we identify the elements v and $(v,0)$ for all $v \in V$. Note, keeping these identifications in mind, that

$$v^{-1}av = a \cdot v, \quad a \in A^+, \ v \in V,$$

where $a \cdot v$ is the product in A of a and v. As is customary, we suppress the dot when writing products in A. We denote the commutator $x^{-1}y^{-1}xy$ of two elements x and y by $[x,y]$. If follows, recalling that we are using additive notation in A^+, that if $a \in A^+$ and $v \in V$, then, in H, we have

$$[a,v] = -a + av = a(v-1).$$

Therefore,

$$H' = gp([a,v]\colon a \in A^+, \ v \in V) = \left\{ \sum a_v(v-1)\colon a_v \in A^+, v \in V \right\},$$

where $\sum$ indicates a finite sum. Letting $U = \{v - 1\colon v \in V\}$, H' is therefore the additive group of the ideal in A generated by U. We express this as

$$H' = AU.$$

We have thus proved the following lemma.

Lemma 4.1. *Let A^+ be the additive group of a commutative ring A with unity, let V be a subgroup of the multiplicative group of units of A, and let $H = A^+ \rtimes V$. Then $H' = AU$, where $U = \{v - 1\colon v \in V\}$.*

For any set U, we let U^n, $n \geq 1$, denote the set of all possible products of n elements in U. If A is a ring we let AU^n, $n \geq 1$, denote the additive group of the ideal in A generated by U^n. We can now verify the following corollary.

Corollary 4.1. *With A, H, and U as in Lemma 4.1, $\gamma_{n+1}H = AU^n$, $n \geq 1$.*

Proof. We use induction on n. The case $n = 1$ is given in Lemma 4.1. Recalling the identification of elements made in the proof of Lemma 4.1, we may write a typical element in $H = A^+ \rtimes V$ as va, where $v \in V$ and $a \in A^+$. Assume $\gamma_n H = AU^{n-1}$. Now $\gamma_{n+1}H = [\gamma_n H, H]$ is generated by all commutators $[m, va]$ with $m \in \gamma_n H$ and $va \in H$. Keeping in mind that $\gamma_n H \le A^+$ and that additive notation is used in A^+, we have

$$[m, va] = -m + mv = m(v-1).$$

And we may write

$$\gamma_{n+1}H = \left\{ \sum m_v(v-1)\colon m_v \in \gamma_n H, v \in V \right\}, \quad n \ge 1,$$

where $\sum$ denotes a finite sum. Applying the induction hypothesis to m_v gives the desired result:

$$\gamma_{n+1}H = \left\{ \sum a_v(v_{i_1}-1)(v_{i_2}-1)\cdots(v_{i_n}-1)\colon a_v \in A^+, v_{i_j} \in V, 1 \le j \le n \right\}$$
$$= AU^n, \quad n \ge 1. \qquad \square$$

Lemma 4.2. *Let A, U, and V be as in Lemma* 4.1. *Let X be a generating set for V. Then the ideal in A generated by $U = \{v - 1\colon v \in V\}$ is the same as the ideal in A generated by $I = \{x - 1\colon x \in X\}$.*

The proof of Lemma 4.2 involves induction on the length of a word in I. The technique is the same as that used by Johnson [4] in his proof about the generating set of an augmentation ideal.

It should be noted that $AU^n = (AU)^n$, $n \ge 1$. This follows from induction and the definition of the product of two ideals. Lemma 4.2 yields $AU = AI$. Therefore,

$$\gamma_{n+1}H = AU^n = (AU)^n = (AI)^n = AI^n, \quad n \ge 1.$$

Our aim is to show that $E = BS^{-1} \rtimes V$ is residually nilpotent. Equivalently, we need to prove that $\bigcap_{r=0}^{\infty} \gamma_{r+1}E = 0$. We recall that V is the group generated by t_i and $f_i = 1 + t_i + t_i^2$, $1 \le i \le n$. Lemmas 4.1 and 4.2 imply that $\gamma_1 E = E'$ is the additive group of the ideal in BS^{-1} generated by

$$\{t_i - 1, f_i - 1\colon 1 \le i \le n\} = \{t_i - 1, t_i(t_i + 1)\colon 1 \le i \le n\}. \tag{4.1}$$

But $t_i BS^{-1} = BS^{-1}$ because t_i is invertible in BS^{-1}, and so (4.1) may be replaced by

$$\{t_i - 1, t_i + 1\colon 1 \le i \le n\}.$$

Equivalently, by subtraction, E' is the additive group of the ideal in BS^{-1} generated by the set

$$I = \{2, t_1 + 1, t_2 + 1, \ldots, t_n + 1\}.$$

That is, $E' = (BS^{-1})I$. From Corollary 4.1, subsequent terms of the lower central series of E are powers of E'. Thus, $\gamma_{r+1}E = (BS^{-1})I^r$ for $r \geq 1$, where

$$I^r = \left\{ 2^{\varepsilon_0}(t_1+1)^{\varepsilon_1}(t_2+1)^{\varepsilon_2}\cdots(t_n+1)^{\varepsilon_n} : \sum_{k=0}^{n} \varepsilon_k = r \right\}. \tag{4.2}$$

As mentioned earlier in Section 4, BS^{-1} is a free $\mathbb{Z}T\cdot S^{-1}$-module. In fact, with B a free $\mathbb{Z}T$-module on $\{a_1, a_2, \ldots, a_n\}$, we have

$$BS^{-1} = \bigoplus_{i=1}^{n} \mathbb{Z}Ta_i \cdot S^{-1}$$

(see, for example, [8]). Thus, by a simple induction,

$$(BS^{-1})I^r = \left\{ \bigoplus_{i=1}^{n} \mathbb{Z}Ta_i \cdot S^{-1} \right\} I^r \simeq \bigoplus_{i=1}^{n} (\mathbb{Z}Ta_i \cdot S^{-1})I^r.$$

Furthermore, it is readily seen that

$$\bigcap_{r=0}^{\infty} \bigoplus_{i=1}^{n} \{(\mathbb{Z}Ta_i \cdot S^{-1})I^r\} = \bigoplus_{i=1}^{n} \bigcap_{r=0}^{\infty} \{(\mathbb{Z}Ta_i \cdot S^{-1})I^r\}. \tag{4.3}$$

Here it is to be understood that $(\mathbb{Z}Ta_i \cdot S^{-1})I^0 = \mathbb{Z}Ta_i \cdot S^{-1}$. Since $\mathbb{Z}Ta_i$ is an isomorphic copy of $\mathbb{Z}T$, it follows from (4.3) that in order to obtain the desired result, namely, $\bigcap_{r=0}^{\infty} \gamma_{r+1}E = 0$, it suffices to verify that $\bigcap_{r=0}^{\infty} (\mathbb{Z}T \cdot S^{-1})I^r = 0$. This is done in Section 5.

5. Proof of the theorem

To simplify the notation, let

$$J^0 = \mathbb{Z}T \cdot S^{-1} \quad \text{and} \quad J^r = (\mathbb{Z}T \cdot S^{-1})I^r, \quad r > 0. \tag{5.1}$$

Also, let

$$J^\omega = \bigcap_{r=0}^{\infty} J^r.$$

As concluded in Section 4, the theorem of this paper (stated in Section 1) is true if $J^\omega = 0$. We first show that an element in J^ω has the value 0 at $t_i = -1$, $1 \leq i \leq n$. Then we verify that all the repeated partial derivatives of an element in J^ω are in J^ω. With every element in J^ω we associate a polynomial factor, also belonging to J^ω. The Taylor series of this polynomial in powers of $t_i + 1$, $1 \leq i \leq n$, then yields the result that $J^\omega = 0$.

We begin by letting

$$\phi : \mathbb{Z}T \cdot S^{-1} \to \mathbb{Z}$$

be evaluation at $t_i = -1$, $1 \leq i \leq n$. ϕ is well-defined since every element of $\mathbb{Z}T \cdot S^{-1}$ exists at $t_i = -1$, $1 \leq i \leq n$. ϕ is a ring homomorphism. And $\mathbb{Z} \subset \mathbb{Z}T \cdot S^{-1}$ implies that ϕ is an epimorphism.

Lemma 5.1. *With ϕ as defined above, $J^{\omega}\phi = 0$.*

Proof. From the generators (4.2) of J^r we have that

$$J^r\phi = (\mathbb{Z}T \cdot S^{-1})I^r\phi = 2^r\mathbb{Z}, \quad r \geq 0.$$

$J^{\omega}\phi = 0$ then follows from

$$J^{\omega}\phi \subseteq \bigcap_{r=0}^{\infty} J^r\phi = \bigcap_{r=0}^{\infty} 2^r\mathbb{Z} = 0. \qquad \square$$

Next we show that all the partial derivatives of a member of J^{ω} are also in J^{ω}. It is evident from the nature of an element $x \in \mathbb{Z}T \cdot S^{-1} = J^0$ that $\partial x/\partial t_q \in \mathbb{Z}T \cdot S^{-1}$, $1 \leq q \leq n$. It follows that $(\partial/\partial t_q)J^0 \subseteq J^0$, $1 \leq q \leq n$. A typical element $x \in (\mathbb{Z}T \cdot S^{-1})I = J$ is expressible as

$$x = 2z_0 + \sum_{i=1}^{n} (t_i + 1)z_i, \quad z_i \in \mathbb{Z}T \cdot S^{-1}, \quad 0 \leq i \leq n.$$

Then, since $\partial z_i/\partial t_q \in \mathbb{Z}T \cdot S^{-1}$, $0 \leq i \leq n$, $1 \leq q \leq n$, we have

$$\begin{aligned} \frac{\partial x}{\partial t_q} &= 2\frac{\partial z_0}{\partial t_q} + \sum_{i=1}^{n} (t_i + 1)\frac{\partial z_i}{\partial t_q} + z_q \\ &\in (\mathbb{Z}T \cdot S^{-1})I + \mathbb{Z}T \cdot S^{-1} = \mathbb{Z}T \cdot S^{-1}, \quad 1 \leq q \leq n. \end{aligned}$$

It follows that $(\partial/\partial t_q)J \subseteq J^0$, $1 \leq q \leq n$. Lemma 5.2 is needed to claim that, in general, $(\partial/\partial t_q)J^r \subseteq J^{r-1}$, $r \geq 1$.

Lemma 5.2. *In a commutative ring R with unity, if U is the ideal generated by $\{u_1, u_2, \ldots, u_k\}$, then*

$$U^n = u_1U^{n-1} + u_2U^{n-1} + \cdots + u_kU^{n-1}, \quad n \geq 1.$$

Proof. We use induction on n. If $n = 1$, we know that

$$U = u_1R + u_2R + \cdots + u_kR,$$

and, conventionally, $R = U^0$. Assume the lemma is true for U^i, $0 < i < n$. It then follows from the distributivity of ideals and the definition of the product of two ideals that

$$\begin{aligned} U^n = UU^{n-1} &= U(u_1U^{n-2} + u_2U^{n-2} + \cdots + u_kU^{n-2}) \\ &= u_1U^{n-1} + u_2U^{n-1} + \cdots + u_kU^{n-1}. \qquad \square \end{aligned}$$

Lemma 5.2 allows us to write

$$J^r = 2J^{r-1} + \sum_{i=1}^{n} (t_i + 1)J^{r-1}, \quad r \geq 1.$$

Then, using the identical argument as for $(\partial/\partial t_q)J \subseteq J^0$, just prior to Lemma 5.2, we have by induction,

$$\frac{\partial}{\partial t_q} J^r \subseteq J^{r-1}, \quad r \geq 0,\ 1 \leq q \leq n. \tag{5.2}$$

Here and in the following discussion it is to be understood that $J^m = J^0$ if $m < 0$. Applying (5.2) to second-order partial derivatives, we have

$$x \in J^r \;\Rightarrow\; \frac{\partial^2 x}{\partial t_p \partial t_q} = \frac{\partial}{\partial t_p}\left(\frac{\partial x}{\partial t_q}\right) \in \frac{\partial}{\partial t_p} J^{r-1} \subseteq J^{r-2}$$

for $r \geq 0$, $1 \leq p, q \leq n$. Inductively,

$$x \in J^r \;\Rightarrow\; \frac{\partial^k x}{\partial t_{i_1} \partial t_{i_2} \ldots \partial t_{i_k}} \in J^{r-k} \quad r \geq 0, k \geq 1, \quad 1 \leq i_1, i_2, \ldots, i_k \leq n. \tag{5.3}$$

Now, from (5.2), we have

$$x \in J^\omega \;\Rightarrow\; x \in J^r, r \geq 0 \;\Rightarrow\; \frac{\partial x}{\partial t_q} \in J^\omega, \quad 1 \leq q \leq n.$$

More generally, (5.3) gives the following lemma.

Lemma 5.3. *If $x \in J^\omega$, then $\partial^k x/\partial t_{i_1} \ldots \partial t_{i_k} \in J^\omega$ for $k \geq 1$, $1 \leq i_1, \ldots, i_k \leq n$.*

Recalling Lemma 5.1, we can now say that if $x \in J^\omega$, evaluation gives

$$x\Big|_{t_j = -1} = 0 \quad \text{and} \quad \frac{\partial^k x}{\partial t_{i_1} \ldots \partial t_{i_k}}\Big|_{t_j = -1} = 0, \quad 1 \leq j \leq n, k \geq 1, \ 1 \leq i_1, \ldots, i_k \leq n.$$

We now introduce a polynomial factor. $x \in J^\omega \Rightarrow x \in \mathbb{Z}T \cdot S^{-1}$, i.e.,

$$x = p(t_1, t_2, \ldots, t_n)\left(\prod_{i=1}^{n} t_i^{j_i}\right)\left(\prod_{i=1}^{n} (1 + t_i + t_i^2)^{k_i}\right), \quad j_i, k_i \in \mathbb{Z}, \tag{5.4}$$

where p is a polynomial in $t_1, t_2, \ldots, t_n$ with integer coefficients. Now, $J^\omega = \bigcap_{r=0}^{\infty} J^r$ is an ideal of $\mathbb{Z}T \cdot S^{-1}$. The element

$$y = \left(\prod_{i=1}^{n} t_i^{j_i}\right)\left(\prod_{i=1}^{n} (1 + t_i + t_i^2)^{k_i}\right)$$

is an invertible element of $\mathbb{Z}T \cdot S^{-1}$. Hence,

$$xy^{-1} = p(t_1, t_2, \ldots, t_n) \in J^\omega.$$

In particular, as found above, p and all its partial derivatives of all orders vanish at $t_i = -1, 1 \le i \le n$. This suggests considering the Taylor series for the polynomial p in powers of $t_i + 1$, $1 \le i \le n$. That is,

$$p(t_1, t_2, \dots, t_n) = p(-1, -1, \dots, -1)$$
$$+ \sum \frac{1}{k!} \left\{ \left[(t_1 + 1)\frac{\partial}{\partial t_1} + \dots + (t_n + 1)\frac{\partial}{\partial t_n} \right]^k p(t_1, t_2, \dots, t_n) \right\}_{\substack{t_i = -1 \\ 1 \le i \le n}}.$$

The powers of the differential operators $[(t_1 + 1)\partial/\partial t_1 + \cdots + (t_n + 1)\partial/\partial t_n]^k$ are found by the usual binomial expansion. Every term is equal to 0 in the above Taylor series for p. Hence,

$$p(t_1, t_2, \dots, t_n) = 0.$$

Recall that $x \in J^\omega$ has the form (5.4). We have therefore achieved our aim, namely, if $x \in J^\omega$ then $x = 0$.

It follows from the discussion in Section 4 that

$$J^\omega = \bigcap_{r=0}^{\infty} J^r = \bigcap_{r=0}^{\infty} (\mathbb{Z}T \cdot S^{-1}) I^r = 0$$

implies that $\bigcap_{r=0}^{\infty} \gamma_{r+1} E = 0$, or E is residually nilpotent. We have therefore verified the theorem stated in Section 1, namely, a finitely generated free metabelian group is a subgroup of a finitely presented metabelian group that is also residually nilpotent.

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